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Dedicated to Jaco de Bakker, at the occasion of his goodbye to CWI

Abstract

This short note starts with some personal words to Jaco de Bakker. They are followed by two problems in recreational mathematics about the length of reduction paths, consisting of partitions of some natural number n. One of these problems came up in an attempt to prove the famous conjecture P = NP. Solutions are given in the Appendix.

Dear Jaco,

we start with some personal words, from each of us.

Wim: we first met in February 1986 when I gave a talk at the CWI on unbounded nondeterminacy of recursive pocedures. Some years later, you encouraged and stimulated me to write a book on that subject for Cambridge University Press, where you were on the Editorial Board.

Gerard: I first met you on 28 April 1989, at the celebration of your 25-year jubilee at CWI. I had entered Computer Science a few years before, with a background in mathematical logic. You were (and still are) for me one of the legendary examples of mathematicians with great impact on computer science. As a logician, I appreciate your groundbreaking work for semantics: it relates computer science in a seminal way to logic and linguistics.

This contribution to your *Liber Amicorum* is about two related problems of a recreational-mathematical nature. The first problem arose from an attempt by A.D. Plotnikov to prove P = NP, via an efficient algorithm for the minimal clique partition problem. The second problem is a mutation of the first, created by an error in copying it. Solutions are given in the Appendix.

Reduction steps in partitions

$P(n)$ denotes the collection of partitions of natural number n. Partitions will be represented by weakly decreasing sequences. So we have, e.g.

$$P(5) = \{5, 41, 32, 311, 221, 2111, 11111\}$$

For reasons of mathematical smoothness, we make these sequences infinite by adding an infinite tail of zero's. The lexicographical order $<_L$ is defined as usual: $a <_L b$

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iff, for some k, $a_k > b_k$ and $a_i = b_i$ for all $i < k$. In addition, we define the *tail ordering* $\geq_T$ by

$$a \geq_T b =_{\text{def}} \forall n\; a_n \geq b_{n+1}$$

i.e. a is pointwise greater than or equal to the tail of b. With $<_L$ and $\geq_T$, we define two reduction relations on $P(n)$:

$$a \to b =_{\text{def}} (a <_L b \wedge a \geq_T b)$$

$$a \Rightarrow b =_{\text{def}} (a <_L b \wedge b \geq_T a)$$

We observe that $\to$ nor $\Rightarrow$ is transitive:

$$1111 \to 211 \to 22, \text{ but } 1111 \not\to 22$$

$$222 \Rightarrow 321 \Rightarrow 411, \text{ but } 222 \not\Rightarrow 411$$

Now let $p(n)$ be the number of partitions of n, $r_1(n)$ the maximal length (i.e. number of nodes) of a $\to$-chain in $P(n)$, and $r_2(n)$ the maximal length of a $\Rightarrow$-chain in $P(n)$. We computed the first values of these functions:

n	1	2	3	4	5	6	7	8	9	10	11
$p(n)$	1	2	3	5	7	11	15	22	30	42	56
$r_1(n)$	1	2	3	5	7	11	15	19	25	33	42
$r_2(n)$	1	2	3	5	7	10	14	19	25	33	43

Observe that, for $n \leq 5$, the one-step lexicographical order in $P(n)$ equals both $\to$ and $\Rightarrow$. In $P(6)$ this symmetry is broken by the fact that $222 \not\Rightarrow 3111$; in $P(8)$, we have $2222 \not\to 3111111$, $332 \not\to 41111$ and $44 \not\to 5111$.

We have $\log(p(n)) = \Theta(\sqrt{n})$ (see [2, p. 278]), so the asymptotic behaviour of $p(n)$ is subexponential. The question is: what is the asymptotic behaviour of r_1 and r_2?

Problem 1. Is the length of the maximal $\to$-reduction path polynomial in n, i.e. is r_1 polynomial?

Problem 2. Is the length of the maximal $\Rightarrow$-reduction path polynomial in n, i.e. is r_2 polynomial?

History. In 2000, the paper *An Efficient Algorithm for the Minimum Clique Partition Problem* by Anatoly D. Plotnikov was available on www via the address `www.busygin.dp.ua/clipat.html`. It claimed to contain a polynomial-time algorithm for the Minimum Clique Partition Problem (MCPP). This is the problem of finding a partition of a finite graph into the minimum number of cliques. (It is equivalent to the Minimum Graph Coloring Problem for the complementary graph.) Since MCPP is NP-hard, this would provide a positive answer on the long-standing open question P =? NP.

In a review of the paper (then available at the same www-address), Stas Busygin wrote:

> In general, the paper correctly describes a new conception on graphs (so-called vertex-saturation) and on the basis of it presents a new NP-complete problem (the MPP problem) heavily related to the minimum clique partition problem. However, the author fails not only to show polynomial time solvability of the MPP problem (that is doubtful after all) but also to clarify its polynomial equivalence to the considered NP-hard problem.

The second author (Hesselink) studied Plotnikov's paper, with the intention to transform the good ideas into correct algorithms (see [1]). In the analysis of the time complexity of one of the subalgorithms, he came upon the definition of the reduction relation $\rightarrow$ and the function r_1, and conjectured that $r_1(n)$ was bounded by n^2. He discussed $\rightarrow$ and r_1 with the first author (Renardel), and the latter tried to answer the question of the polynomiality of r_1. However, in doing so he inadvertedly transformed it into another problem, viz. the polynomiality of r_2 based on $\Rightarrow$! So when he presented the solution to Hesselink, it became clear that the wrong problem had been solved. Renardel tried again, and finally succeeded in solving the original problem, too.

In the meantime, Plotnikov's paper mentioned above is no longer available at www. The address `www.busygin.dp.ua/clipat.html` now (May 2002) refers to a thorough revision of the paper, under the title *Four NP-Complete Problems.* From Busygin's comment (at the same address), we quote:

> Now, after about 1.5 years, the author has reworked a significant part of his manuscript – where the flaws had been found – and extended the proposal to the maximum independent set problem as well. Definitely, the publication should NOT be taken as the final P=NP conclusion and each of its claims is subject to a thorough questioning and discussion. Moreover, I completely understand those who will abstain from spending own time and energy on analyzing the proposal on the basis of its probable falseness. [...] However, I do consider the work of Anatoly Plotnikov on the famous NP-hard graph problems interesting enough to be presented here and invite everyone working in the computational complexity field to participate in its investigation.

References

[1] W.H. Hesselink, *The borderline between P and NP.* To be obtained from `www.cs.rug.nl/~wim/pub/mans.html`

[2] H. Rademacher, *Topics in Analytic Number Theory.* Springer-Verlag, Berlin, 1973.

Appendix

Solution of Problem 1

We claim that, for all $n \geq 1$:

there is a reduction path of length 2^n in $P((3n^2+n)/2)$

So $r_1(n)$ is not polynomial: like $p(n)$, it is of order $2^{\sqrt{n}}$.

The construction of the reduction path is based on three ideas. Firstly, we observe that the path consisting of the first 2^n binary numbers $0\ldots0$ to $1\ldots1$ is in lexicographical order. For $n=3$:

000 001 010 011 100 101 110 111

However, these sequences are not weakly decreasing and the path is not in tail order. We apply the second idea: add, to every sequence in the path, pointwise the sequence $(n-1)\ (n-2)\ \ldots\ 1\ 0$. The resulting sequences are weakly decreasing, and the resulting path is in lexicographical order and in tail order, e.g.:

210 211 220 221 310 311 320 321

But the sequences in the path do not represent partitions of the same number. The third idea resolves this: add, at the end of every sequence in the path, the number of zeroes of the binary number that was used in its definition; moreover, add n to the other numbers in the sequence in order to keep it weakly decreasing. This leads to a $\rightarrow$-reduction path, e.g. for $n = 3$ in $P(15)$:

$$5433 \rightarrow 5442 \rightarrow 5532 \rightarrow 5541 \rightarrow 6432 \rightarrow 6441 \rightarrow 6531 \rightarrow 654$$

In the general case, each sequence is a partition of $(\Sigma_{i=0}^{n-1} i) + n^2 + n = 3(n^2 + n)/2$.

Solution of Problem 2

We claim that, for all n:

> there is a $\Rightarrow$-reduction path of length 2^n in $P((n^2 + n + 2)/2)$

So $r_2(n)$ is not polynomial: like $p(n)$, it is of order $2^{\sqrt{n}}$.

We find it convenient to reverse the order here and to work with $\Leftarrow$. The proof is based on the following observation

$$x\ (m+1)\ 1^p \Leftarrow x\ m\ m\ 1^{p-m+1}$$

where x is some finite weakly decreasing sequence of numbers $\geq m+1$, and 1^p denotes the sequence of p 1's. With induction over m we can prove:

> if k is sufficiently large, then there is a $\Leftarrow$-reduction path of length $2^m + 1$ from $x\ (m+2)\ i^k$ to $x\ (m+1)\ 1^{k+1}$ (1)

The induction step is proved by first using

$$x\ (m+2)\ 1^k \Leftarrow x\ (m+1)\ (m+1)\ 1^{k-m}$$

followed by applying the induction hypothesis $m-1$ times, on $(x\ (m+1)\ p\ 1^{k-p+1})$, where $m+1 \geq p \geq 2$. Concatenate the reduction paths obtained this way, and we get the desired reduction path. For an example, we take x empty, $m = 2$ and $k = 3$:

$$4111 \Leftarrow 331 \Leftarrow 322 \Leftarrow 3211 \Leftarrow 31111$$

Now apply (1) n times with x the empty sequence and m running from $n-1$ to 0, and concatenate the reduction paths. This yields a reduction path of length 2^n from $(n+1)\ 1^k$ to 1^{k+n+1}, and we only have to determine the 'sufficiently large' value of k for which this all works well in $P(k+m+1)$. It is quite obvious that the critical position in the path, where the exponent of 1 takes its lowest value, is the sequence $n\ (n-1)\ (n-2) \ldots 3\ 2\ 2\ 1^{k-(n^2-n)/2}$. So $k = (n^2 - n)/2$ is just large enough: then all sequences are in $P(k+n+1) = P((n^2+n+2)/2)$. For $n = 3$ this yields the reduction path

$$4111 \Leftarrow 331 \Leftarrow 322 \Leftarrow 3211 \Leftarrow 31111 \Leftarrow 22111 \Leftarrow 211111 \Leftarrow 1111111$$